On a lower bound of the number of integers in Littlewood's conjecture (arXiv: 2207.13462, 2401.05027)

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Littlewood's conjecture

For $x \in \mathbb{R}$, we write $\langle x \rangle = \min_{k \in \mathbb{Z}} |x - k|$. It is known that

$$\liminf_{n \to \infty} n \langle n\alpha \rangle = 0 \iff \forall \varepsilon > 0, \exists m/n \in \mathbb{Q}, \left| \alpha - \frac{m}{n} \right| \leq \frac{\varepsilon}{n^2}$$

holds for Lebesgue a.e. α (Khinchine's theorem), but

$$\mathbf{Bad} = \left\{ \alpha \in \mathbb{R} \,\middle|\, \liminf_{n \to \infty} n \langle n\alpha \rangle > 0 \right\}$$

has full Hausdorff dimension ([Jarník, 1928]).

Littlewood's conjecture (c.1930).

For every $(\alpha, \beta) \in \mathbb{R}^2$,

$$\liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle = 0,$$

Results toward Littlewood's conjecture

- Littlewood's conjecture is trivial when α and β belong to the same square number field.
  [Cassels, Swinnerton-Dyer, 1955]. Littlewood's conjecture is true when α and β belong to the same <u>cubic</u> number fields.

- [Pollington, Velani, 2000]. For $\forall\alpha \in \mathbf{Bad}$, $\exists \mathbf{G}(\alpha) \subset \mathbf{Bad}$ with $\dim_H \mathbf{G}(\alpha) = 1$ s.t., if $\beta \in \mathbf{G}(\alpha)$, then

$$n\langle n\alpha\rangle\langle n\beta\rangle \leq \frac{1}{\log n} \quad \text{for infinitely many } n \in \mathbb{N}.$$

The set of exceptions to Littlewood's conjecture has Hausdorff dimension zero.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

$$\dim_H \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle > 0 \right\} = 0.$$

Furthermore, this set is an at most countable union of compact sets of box dimension zero.

This Theorem is obtained as a corollary of some property of the **diagonal action on** $\mathrm{SL}(3, \mathbb{R}) / \mathrm{SL}(3, \mathbb{Z})$.

The diagonal action on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$

We write

$$G := \mathrm{SL}(3,\mathbb{R}),\ \Gamma := \mathrm{SL}(3,\mathbb{Z}),\ X := G/\Gamma.$$

$X = G/\Gamma$ admits a unique G-invariant Borel probability measure m_X on X, called the **Haar measure**. However, X is *not compact*.
Let

$$A := \left\{ \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{pmatrix} \middle| \ t_1, t_2, t_3 \in \mathbb{R},\ t_1 + t_2 + t_3 = 0 \right\} < G.$$

The left action of A

$$A \times X \ni (a, x) \mapsto ax \in X$$

is called **the (higher rank) diagonal action** on X.

The relation between the diagonal action and Littlewood's conjecture

We define the positive cone A^+ of A by

$$A^+ := \left\{ a_{s,t} := \begin{pmatrix} e^{-s-t} & & \\ & e^s & \\ & & e^t \end{pmatrix} \middle| \, s, t \geq 0 \right\}.$$

For $(\alpha, \beta) \in \mathbb{R}^2$, we write

$$u_{\alpha,\beta} := \begin{pmatrix} 1 & & \\ \alpha & 1 & \\ \beta & & 1 \end{pmatrix} \in G, \quad \tau_{\alpha,\beta} = u_{\alpha,\beta}\Gamma \in X.$$

Key Proposition.

For $(\alpha, \beta) \in \mathbb{R}^2$, $\liminf_{n\to\infty} n\langle n\alpha \rangle \langle n\beta \rangle = 0$ iff the A^+ orbit of $\tau_{\alpha,\beta}$ is unbounded in X.

Measure rigidity under positive entropy condition

For an A-invariant probability measure μ and $a \in A$, we write $h_\mu(a)$ for the entropy of the map $X \ni x \mapsto ax \in X$ w.r.t. μ.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

If μ is an A-invariant and ergodic Borel probability measure on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ s.t. $h_\mu(a) > 0$ for $\exists a \in A$, then μ is the Haar measure m_X on X.

As a corollary of this Theorem, we obtain that

$$\dim_H \left\{ (\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, A^+\tau_{\alpha,\beta} \subset X \text{ is bounded} \right\} = 0$$

(needs more ergodic-theoretic argument). By Key Proposition, this is equivalent to

$$\dim_H \left\{ (\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, \liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle > 0 \right\} = 0.$$

Remarks on measure rigidity

- The similar measure rigidity holds for $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z}), n \geq 3$, but not for $n = 2$.
- The positive entropy condition is believed to be dropped.

Full measure rigidity conjecture [Margulis].

For $n \geq 3$, every A-invariant and ergodic Borel probability measure on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ is homogeneous.

It is known that if Full measure rigidity conjecture is true, then Littlewood's conjecture follows from it.

Quantitative version of Littlewood's conjecture and Main Theorem

Littlewood's conjecture says that, for every $(\alpha, \beta) \in \mathbb{R}^2$ and any $0 < \varepsilon < 1$, $n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon$ for infinitely many n.

Problem (Quantitative ver. of Littlewood's conjecture).

For $(\alpha, \beta) \in \mathbb{R}^2$, $0 < \varepsilon < 1$ and sufficiently large $N \in \mathbb{N}$, **how many integers** $n \in [1, N]$ are there s.t.

$$n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon \quad ?$$

We want to know a lower bound of $|\{n \in [1, N] \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}|$ which is valid for as many (α, β) as possible.

Main Theorem [U., 2022+, 2024+].

For $0 < \forall \gamma < 1/72$, there exists an "exceptional set" $Z(\gamma) \subset \mathbb{R}^2$ with $\dim_H Z(\gamma) \le 90\sqrt{2\gamma}$ s.t., for $\forall (\alpha, \beta) \in \mathbb{R}^2 \setminus Z(\gamma)$ and $0 < \forall \varepsilon < 4^{-1} e^{-2}$,

$$\liminf_{N \to \infty} \frac{(\log \log N)^2}{(\log N)^2} \left| \{ n \in [1, N] \mid n \langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \} \right| \ge \gamma.$$

Corollary.

There exists an "exceptional set" $Z \subset \mathbb{R}^2$ with $\dim_H Z = 0$ s.t., for $\forall (\alpha, \beta) \in \mathbb{R}^2 \setminus Z$ and $0 < \forall \varepsilon < 4^{-1} e^{-2}$,

$$\liminf_{N \to \infty} \frac{(\log \log N)^2}{(\log N)^2} \left| \{ n \in [1, N] \mid n \langle n\alpha \rangle \langle n\beta \rangle < \varepsilon \} \right| \ge C_{\alpha,\beta},$$

where $C_{\alpha,\beta} > 0$ is a constant depending only on (α, β).

About the proof of Main Theorem

For $x \in X$ and $T > 0$, **the T-empirical measure of x w.r.t. A^+** is a probability measure on X defined by

$$\delta^T_{A^+,x} := \frac{1}{T^2} \int_{[0,T]^2} \delta_{a_{s,t}x} \, dsdt.$$

Let $(T_k)_{k=1}^{\infty} \subset \mathbb{R}_{>0}$ be a sequence such that $T_k \to \infty$. If $(\delta^{T_k}_{A^+,x})_{k=1}^{\infty}$ converges to a measure μ on X as $k \to \infty$, (w.r.t. the weak*-topology), then μ is A-invariant but it may be that $\mu(X) < 1$ (since X is not compact.)

If $\mu(X) \leq 1 - \gamma$ for $0 < \gamma \leq 1$, we say that $(\delta^{T_k}_{A^+,x})_{k=1}^{\infty}$ exhibits γ-**escape of mass**.

For $(\alpha, \beta) \in \mathbb{R}^2$, we consider the empirical measures of $x = \tau_{\alpha,\beta}$. Assume that $(\delta^{T_k}_{A^+, \tau_{\alpha,\beta}})_{k=1}^{\infty}$ converges to a measure μ. Let $\gamma > 0$.

Case 1: If μ has the **large entropy**, that is,

$$1 - \gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\mu(X)^{-1}\mu}(a_1) > \gamma,$$

then, by the measure rigidity, a large part of μ consists of the Haar measure m_X.

Case 2: If γ-**escape of mass** occurs, that is, $\mu(X) \leq 1 - \gamma$, then, the A^+ orbit of $\tau_{\alpha,\beta}$ stays close to infinity for a long time.

In these two cases, we can see that

$$\liminf_{k\to\infty} \frac{(\log\log N_k)^2}{(\log N_k)^2} \left|\{n < N_k \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}\right| \geq \frac{\gamma}{72}$$

for $N_k = e^{2T_k}$.

The remained case is when

$$1-\gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\mu(X)^{-1}\mu}(a_1) \leq \gamma.$$

We can show that **this case occurs only on a set of very small Hausdorff dimension**.

Theorem (Hausdorff dimension of the exceptional set).

Let $0 < \gamma < 1$. We write $Z(\gamma)$ for the set of $(\alpha, \beta) \in [0,1]^2$ s.t. $\delta^T_{A^+, \tau_{\alpha,\beta}}$ $(T > 0)$ accumulate to some A-invariant measure μ on X s.t.

$$1-\gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\mu(X)^{-1}\mu}(a_1) \leq \gamma.$$

Then we have

$$\dim_H Z(\gamma) \leq 15\sqrt{\gamma}.$$

Our exceptional set in Main Theorem corresponds to $Z(72\gamma)$.