Multifractal analysis of a parametrized family of von Koch functions

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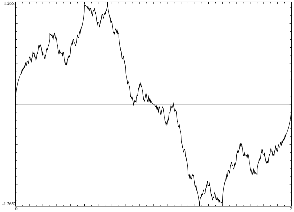
Search for continuous but nowhere differentiable functions :

Riemann Fourier series :

$$\sum_{n\geq 1} \frac{\sin(\pi n^2 x)}{n^2}$$



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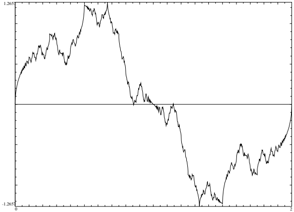
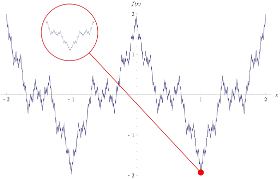


Weierstrass function :

$$0 < H < 1, \quad \sum_{n\geq 1} 2^{-nH} \sin(2^n x)$$



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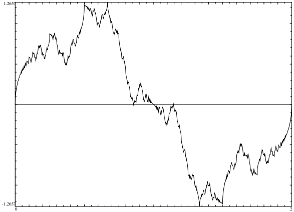

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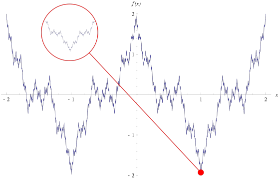

Later, examples by Takagi, Bouligand, ... then Brownian motion....

1- The von Koch Function

(1870–1924)

ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK.

BAND 1.

Sur une courbe continue sans tangente obtenue par une construction géométrique élémentaire

Par HELGE VON KOCH.

Avec 5 figures dans le texte.

Communiqué le 12 Octobre 1904 par G. MITTAG-LEFFLER et KARL BOHLIN.

Jusqu'à l'époque où WEIERSTRASS inventa une fonction continue ne possédant, pour aucune valeur de la variable, une dérivée déterminée[1], c'était une opinion bien répandue dans le monde scientifique que toute courbe continue possède une tangente déterminée (du moins en exceptant certains points singuliers); et l'on sait que, de temps en temps, plusieurs géomètres éminents ont essayé de consolider cette opinion, fondée sans doute sur la représentation graphique des courbes, par des raisonnements logiques.[2]

Bien que l'exemple dû à WEIERSTRASS ait pour toujours corrigé cette erreur, il me semble que cet exemple ne satisfait pas l'esprit au point de vue géométrique; car la fonction dont il s'agit est définie par une expression analytique qui cache la nature géométrique de la courbe correspondante de sorte qu'on ne voit pas, en se plaçant à ce point de vue, pourquoi la courbe n'a pas de tangente; on dirait plutôt que l'appa-

[1] Voir Journal f. Math., t. 79 (1875).

[2] Parmi ces tentatives nous citerons celles d'AMPÈRE (J. éc. pol. cah. 13), de BERTRAND (Traité de C. diff. et intégr.; t. 1) et de GILBERT (Brux. mém. 8°, t. 23 (1872)). — On trouve des notices historiques et bibliographiques dans l'ouvrage de M. E. PASCAL: Esercisi e note crit. di calcole infinitesimale p. 85—128. Milano 1895. — Voir aussi Encyklopädie der Math. Wiss. II. A. 2, p. 63 et l'ouvrage de M. DINI (traduction LÜROTH-SCHEPP): Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse, p. 88 suiv., p. 205—229.

120 years ago : the von Koch Function

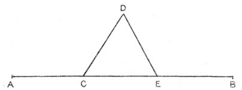

Fig. 1.

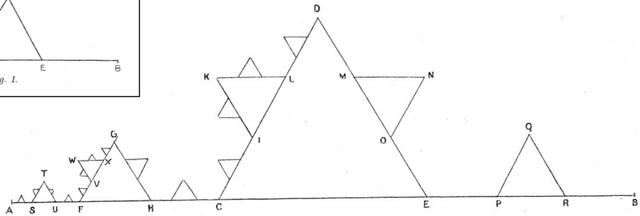

Fig. 2.

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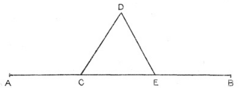

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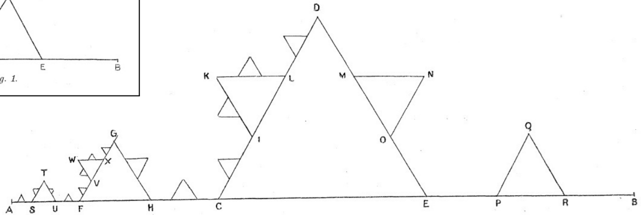

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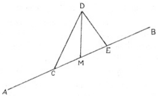

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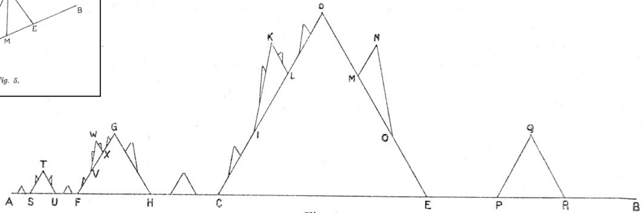
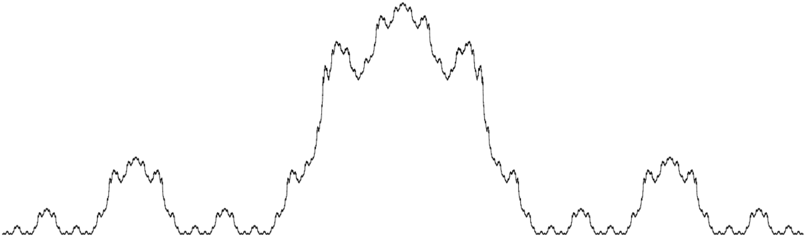
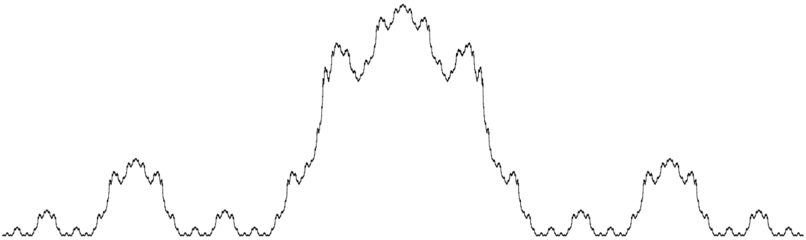


Fig. 4.



Theorem (von Koch, 1904, 1906)

F is a continuous but nowhere differentiable function.



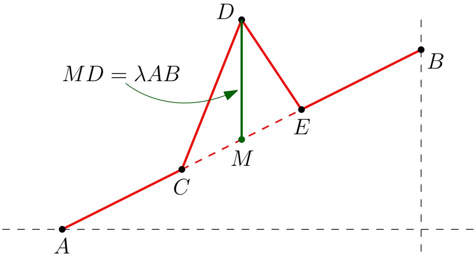
Theorem (von Koch, 1904, 1906)

F is a continuous but nowhere differentiable function.

Question : Multifractal properties of F ?

- Pointwise Hölder exponent ?
- Multifractal spectrum ? Multifractal formalism ?
- Existence of infinite derivatives ?

Fix a parameter parameter $\lambda > 0$,
and consider the following
geometric construction rule :



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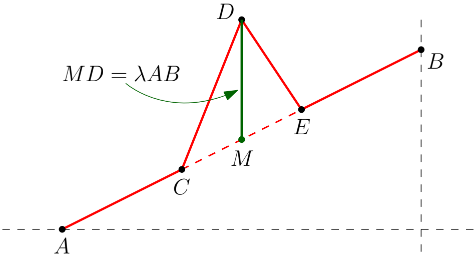
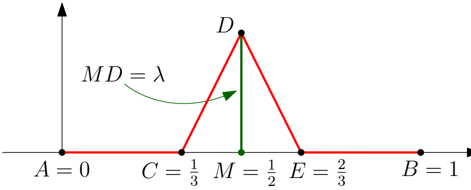


• For all $x \in [0,1]$, $F_0(x) = 0$.

Apply the process to $[0,1]$ to get F_1.



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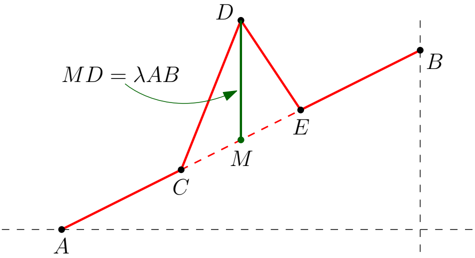

- For all $x \in [0,1]$, $F_0(x) = 0$.

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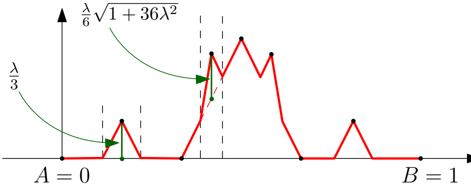
- F_1 is piecewise linear,

Apply the rule to each of the line segments to get F_2.



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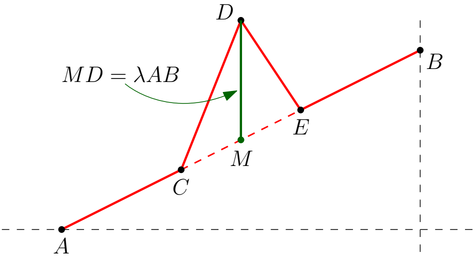

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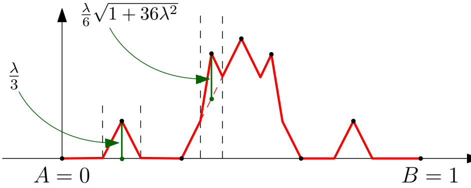
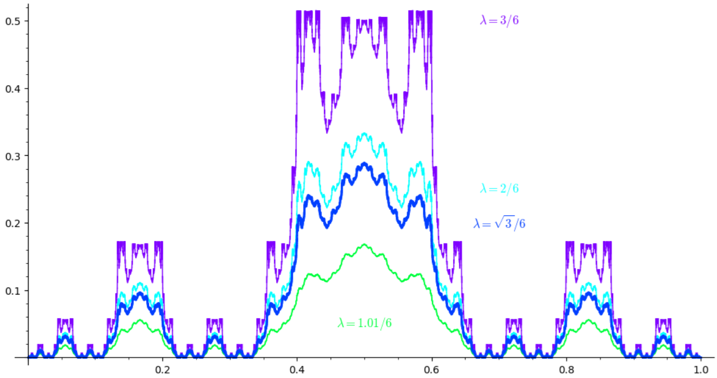
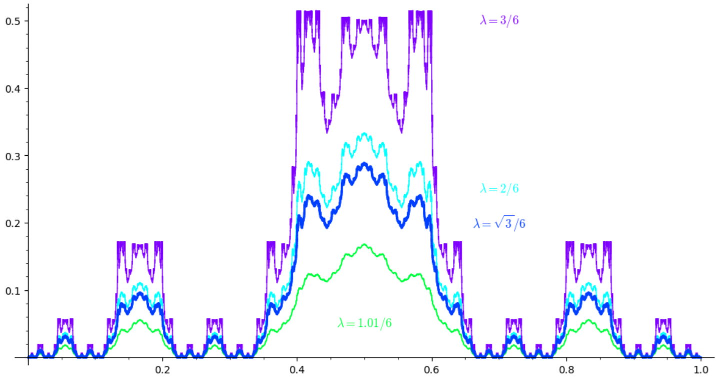


And so on... to get a function $F^\lambda : [0,1] \to \mathbb{R}^+$



0.5
0.4
0.3
0.2
0.1
0.2
0.4
0.6
0.8
1.0
$\lambda = 3/6$
$\lambda = 2/6$
$\lambda = \sqrt{3}/6$
$\lambda = 1.01/6$



- No closed-form expression for $F^\lambda(x)$
- Not a De Rham curve nor a solution to simple functional equations
- Self-similarity or IFS methods cannot directly be used
- The von Koch function corresponds to $\lambda = \frac{\sqrt{3}}{6} \approx 0.289$
- Different ranges of parameters : $(0, \frac{1}{6}]$, $(\frac{1}{6}, \frac{\sqrt{2}}{6}]$, $(\frac{\sqrt{2}}{6}, \frac{1}{3}]$, $(\frac{1}{3}, \frac{5}{6})$, $\geq \frac{5}{6}$.

Pointwise regularity of a function :

In our case all exponents are less than one, so :

Definition

The pointwise Hölder exponent of a locally bounded function f at x_0 is

$$H_f(x_0) = \liminf_{x \to x_0} \frac{\log |f(x) - f(x_0)|}{\log |x - x_0|}.$$

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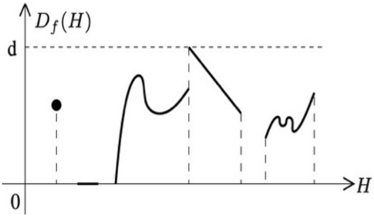
The multifractal spectrum D_f of $f : \mathbb{R}^d \to \mathbb{R}$ is the mapping

$$D_f \, : \, H \longmapsto \dim E_f(H), \quad \text{where} \quad E_f(H) = \{x \in \mathbb{R}^d : H_f(x) = H\}.$$

- dim = Hausdorff
- $\dim \emptyset = -\infty$



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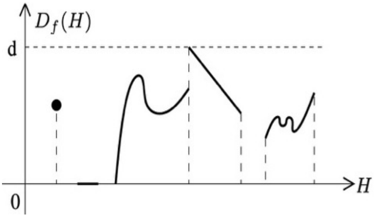
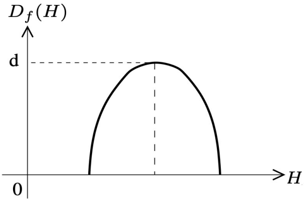
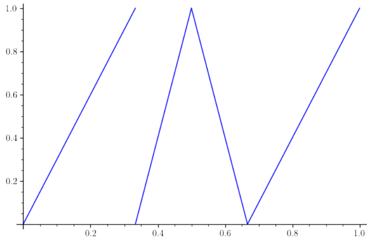




Self-similar measures :

Consider the mapping

$$T(x) = \begin{cases} 3x & \text{if } 0 \le x < \frac{1}{3} \\ 6x - 2 & \text{if } \frac{1}{3} \le x < \frac{1}{2} \\ 4 - 6x & \text{if } \frac{1}{2} \le x < \frac{2}{3} \\ 3x - 2 & \text{if } \frac{2}{3} \le x < 1. \end{cases}$$



Consider the 4 inverse branches of T :
$S_0(x) = \frac{x}{3}$, $S_1(x) = \frac{x}{6} + \frac{1}{3}$,
$S_2(x) = -\frac{x}{6} + \frac{2}{3}$ and $S_3(x) = \frac{x}{3} + \frac{2}{3}$.

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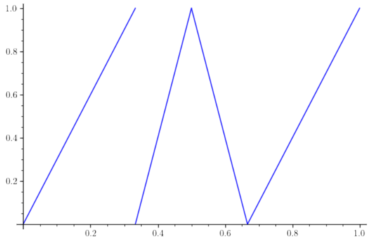

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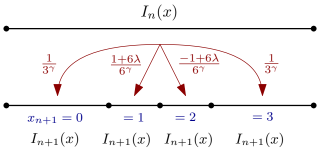
- The attractor of $S = (S_i)_{i=0,1,2,3}$ is $[0,1]$
- S satisfies the OSC
- For $(p_i)_{i=0,\dots,3} \in [0,1]^4$ a probability vector, consider the invariant measure

$$\mu = \sum_{i=0}^{3} p_i \mu \circ S_i^{-1}$$



- The multifractal spectrum d_{μ_λ} of such a self-similar measure μ is known.

Theorem

Let $\lambda \in (\frac{\sqrt{2}}{6}, \frac{5}{6})$*, and let* $\gamma_\lambda \geq 1$ *be the unique real number such that*

$$\frac{1}{3^{\gamma_\lambda}} + \frac{6\lambda+1}{6^{\gamma_\lambda}} + \frac{6\lambda-1}{6^{\gamma_\lambda}} + \frac{1}{3^{\gamma_\lambda}} = 1.$$

Consider the self-similar measure μ_λ *associated with the* $(p_{i,\lambda})_{i=0,\dots,3}$*, where*

$$p_{0,\lambda} = p_{3,\lambda} = \frac{1}{3^{\gamma_\lambda}}, \ p_{1,\lambda} = \frac{6\lambda+1}{6^{\gamma_\lambda}}, \ p_{2,\lambda} = \frac{6\lambda-1}{6^{\gamma_\lambda}}.$$

Then :

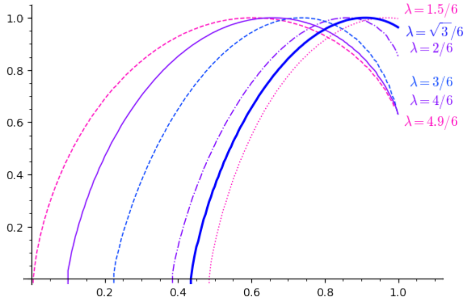
(i) *The support of* $d_{F^\lambda}(\alpha)$ *is* $[\alpha_{\lambda,\min}, 1] := \left[1 - \frac{\log(6\lambda+1)}{\log 6}, 1\right]$.

(ii) *For every* $\alpha \in \left[\alpha_{\lambda,\min}, 1\right]$,

$$d_{F_\lambda}(\alpha) = d_{\mu_\lambda}(\alpha - 1 + \gamma_\lambda).$$

In particular, d_{F_λ} *is strictly concave on its support, the maximum of* d_{F_λ} *is 1 and is reached at the exponent*

$$\alpha_{\lambda,\mathcal{L}} = 1 - \frac{\log(36\lambda^2 - 1)}{4\log 3 + 2\log 6}.$$



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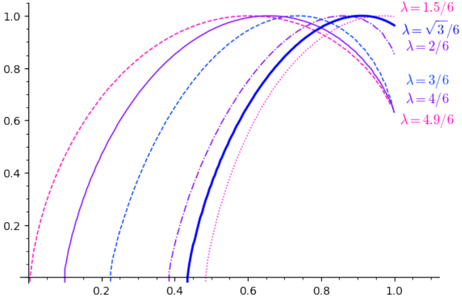
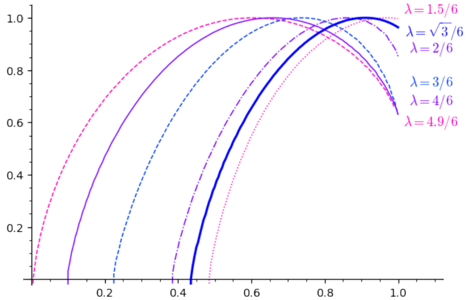


Simple statement for the theorem, but various situations :

- Always concave, touches 0 on the left, but always $\geq \frac{\log 2}{\log 3}$ at 1. Why ?



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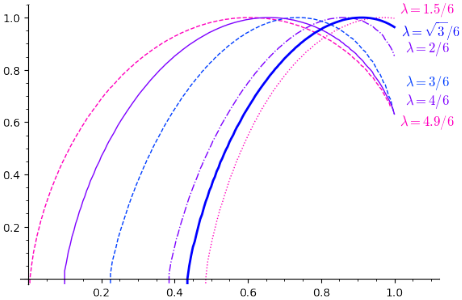

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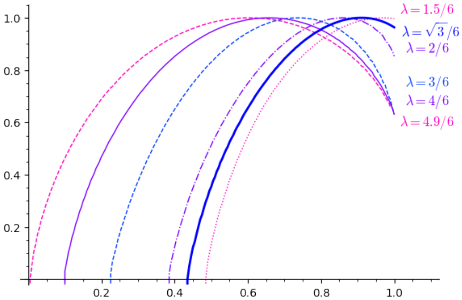

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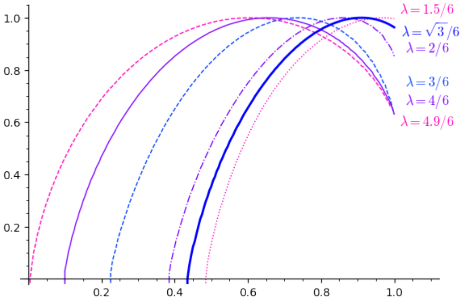
  This corresponds to the situation where $\frac{\log p_{0,\lambda}}{\log 3} = \frac{\log p_{2,\lambda}}{\log 6}$.



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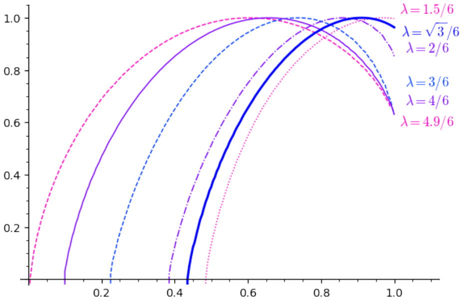
  The value of the spectrum at 1 for $\lambda = \frac{1}{3}$ is the solution to :

$$6^{-s} + 2 \cdot 3^{-s} = 1.$$



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- For $\frac{1}{6} < \lambda \leq \frac{1}{3}$, the graph of the spectrum d_{F^λ} is a continuous mapping of λ.

Theorem

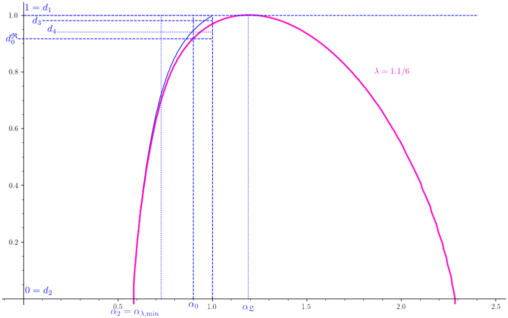
Let $\lambda \in (\frac{1}{6}, \frac{\sqrt{2}}{6}]$, *consider the same quantities* γ_λ, $(p_{i,\lambda})_{i=0,\ldots,3}$ *and the Bernoulli measure* μ_λ *as in the previous theorem.*

(i) *The support of* d_{F^λ} *is* $[\alpha_{\lambda,\min}, 1] = \left[1 - \frac{\log(6\lambda+1)}{\log 6}, 1\right]$.

(ii) *For every* $\alpha \in \left[1 - \frac{\log(6\lambda+1)}{\log 6}, 1\right]$,

$$d_{F^\lambda}(\alpha) \geq d_{\mu_\lambda}(\alpha + 1 - \gamma_\lambda).$$

(ii) *The multifractal spectrum* d_{F^λ} *is continuous at 1.*



The lower bound is obviously not sharp!

Ideas :

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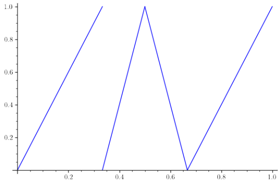

Call $x(n) = 0, 1, 2$ or 3 according to the fact that $T^n(x)$ belongs to the interval $[0, 1/3)$, $(1/3, 1/2)$, $(1/2, 2/3)$ or $(2/3, 1)$.

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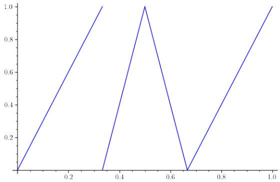
(1) The local regularity of F^λ at x is governed by the orbit $(T^n x)_{n\geq 0}$ of x under T.



Call $x(n) = 0, 1, 2$ or 3 according to the fact that $T^n(x)$ belongs to the interval $[0, 1/3)$, $(1/3, 1/2)$, $(1/2, 2/3)$ or $(2/3, 1)$.

For $\varepsilon_0, \varepsilon_1, ..., \varepsilon_n \in \{0, 1, 2, 3\}^{n+1}$, call

$I_{\varepsilon_1,...,\varepsilon_n} = \{x : x(i) = \varepsilon_i \text{ for } i \in \{0, ..., n\}\}$.

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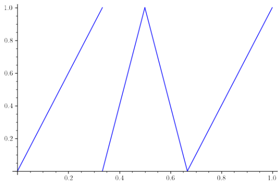

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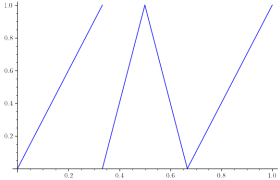

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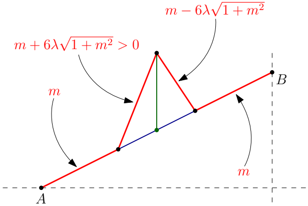
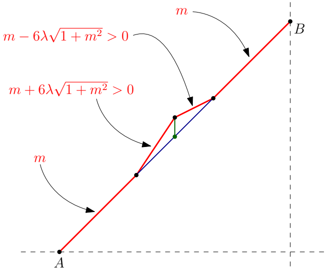
(2) Given x, the slopes on the intermediate functions F_n^λ on $I_n(x)$ are key.



Slope $AB = m > 0$



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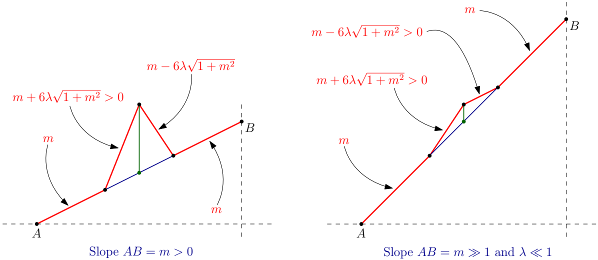

- (Monotonic slopes) Take $\lambda > \frac{1}{6}$ to ensure that $m - 6\lambda\sqrt{1+m^2} < 0$
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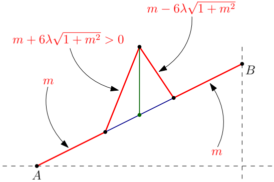

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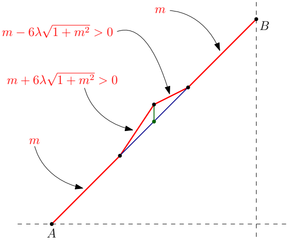

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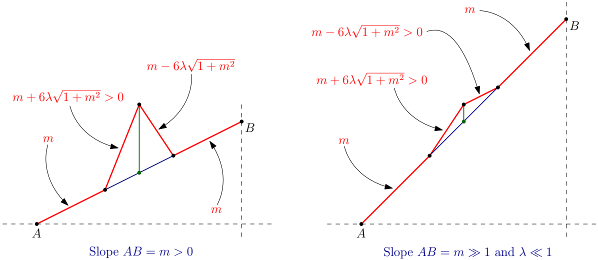

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So, in an perfect world, $m_n(x) \sim \mu_\lambda(I_n(x))|I_n(x)|^{-\gamma_\lambda}$.

Ideas :

(3) Easier when slopes tend to infinity :

Proposition

Provided that $\underline{\dim}(\mu, x) + 1 - \gamma_\lambda \leq 1$ *and* $m_n(x) \to +\infty$,

$$H_{F^\lambda}(x) = \underline{\dim}(\mu, x) + 1 - \gamma_\lambda.$$

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Our approach consists in assuming that points such that $m_n(x) \not\to +\infty$ cannot modify the spectrum.

Proceeding toward a contradiction, we assume that this is the case.

Then we build an IFS satisfying the SOSC on which $m_n(x) \to +\infty$ but with a too large dimension.

Open questions :

- spectrum when $\lambda \in (1/6, \frac{\sqrt{2}}{6}]$?
- spectrum when $\lambda \leq 1/6$?
- multifractal formalism ?
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Danke schön !