L^2 flattening in Fourier decay

Tuomas Sahlsten

University of Helsinki

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joint with **Simon Baker** (Loughborough) and **Osama Khalil** (Illinois Chicago)

Fourier decay

- **Fourier transform** of a measure μ on $\mathbb{R}^d$ is

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} \, d\mu(x), \quad \xi \in \mathbb{R}^d$$

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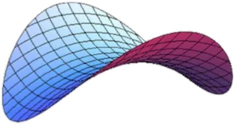
- **Fourier decay**, i.e. decay rates of $\widehat{\mu}(\xi)$ for large $|\xi|$ appear as a tool to study e.g:
  - multiplicity of trigonometric series on fractals
  - restriction theorems on fractals
  - equidistribution and Diophantine properties of fractals
  - patterns on fractals and intersections of fractals
  - absolute continuity and projection properties of fractal measures
  - exponential mixing rates in dynamics
  - fractal uncertainty principles in quantum chaos
  - distribution of scattering resonances of the Laplacian and resulting energy decay rates of wave equations on open hyperbolic systems
- What **techniques** there to study the Fourier decay of μ?

Techniques to study Fourier decay of μ

1. if μ has **"curvature"**, e.g.
   - μ supported on a **curved submanifold of** $\mathbb{R}^d$

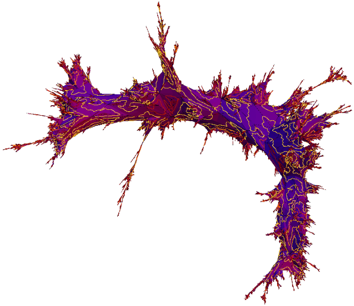
Tools: integration by parts (Van den Corput lemma, stationary phase), for L^6 norms of $\widehat{\mu}$: decoupling techniques (Bourgain, Demeter, Dasu) and Furstenberg set problem (Orponen, Puliatti, Pyörälä)



Techniques to study Fourier decay of μ

2. μ is **"random"**: e.g.
   - **random Cantor construction** (Salem),
   - push-forward under **Brownian motion** and **Levy processes** or level- or graph measures (Kahane, Shieh, Xiao, Fouche, Mukeru, Fraser, Orponen, S., Dysthe, Lai,...)
   - **Liouville Quantum Gravity** (Falconer, Jin), **SI-martingales** (Shmerkin, Suomala), **Gaussian multiplicative chaos** (Garban, Vargas), **Mandelbrot multiplicative cascades** (Chen, Li, Suomala)

Tools: stochastic autosimilarity, time-independence or decay of correlations



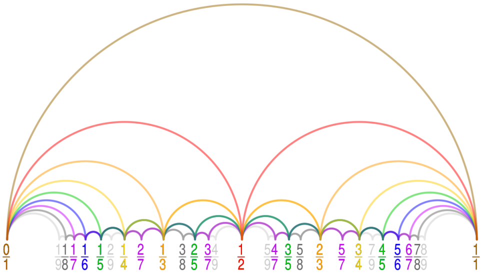
Techniques to study Fourier decay of μ

3. μ has **"Diophantine properties"**, e.g.

• supported on **well-**, **badly-**, **exact approximable** or **Liouville** numbers or vectors (Kaufman, Bluhm, Queffelec, Ramare, Hambrook, Yu, Fraser, Wheeler,...)

• supported on sets of **inhomogeneous approximation** (Hambrook, Chow, Zafeiropoulos, Zorin)

Tools: symmetries/Diophantine properties of continuants, strong uniform non-integrability of the Gauss map, Ostrowski expansions

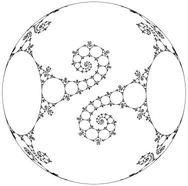
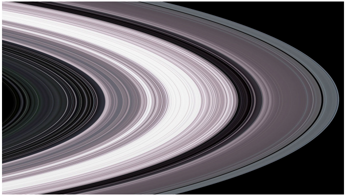


Techniques to study Fourier decay of μ

4. μ is "**stationary for a random walk or dynamical system**", e.g.

- **Patterson-Sullivan measures** of convex cocompact hyperbolic manifolds (Bourgain, Dyatlov, Li, Naud, Pan, Khalil, Baker, S. etc.)
- **Furstenberg- and stationary measures** (Li, Dinh, Kaufmann, Wu)
- **self-similar, conformal or affine measures** (Salem, Erdös, Kahane, Strichartz, Tsujii, Solomyak, Jordan, Lindenstrauss, Varjú, Mosquera, Shmerkin, Li, S., Bremont, Yu, Algom, Hertz, Wang, Baker, Banaji, Chang, Wu, Wu, Rapaport, Streck, Paukkonen, Avila, Lyubich, Zhang, ...)
- **equilibrium measures for Axiom A diffeomorphisms** (Leclerc)

Tools: algebraic properties of contractions (Pisot or not, etc.), spectral theory of transfer operators, representation theory, CLT, renewal theory, discretised sum-product theory, autosimilarity of the temporal distances

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- ...but **Tsujii** (2012) proved $\forall \varepsilon > 0, \exists \kappa = \kappa(\varepsilon) > 0$ s.t. for all $T > 1$:

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  is covered by $C_\varepsilon T^\varepsilon$ intervals $[n, n+1) \subset [-T, T]$, $n \in \mathbb{Z}$.
- Tsujii's result works for self-similar measures, but it is manifestation of:

Average Fourier decay for non-concentrated measures (Khalil, 2023)

If μ on $\mathbb{R}^d$ is "(C, α)-**uniformly affinely non-concentrated**", i.e. $\exists C, \alpha > 0$ s.t. $\forall \varepsilon > 0$, $x \in \operatorname{supp}(\mu)$, $0 < r < 1$ and affine hyperplane W:

$$\mu(y \in B(x, r) : d(y, W) \leq \varepsilon r) \leq C\varepsilon^\alpha \mu(B(x, r)),$$

then: $\forall \varepsilon > 0, \exists \kappa = \kappa(\varepsilon) > 0$ s.t. for all $T > 1$:

$$\{\varrho \in [-T, T]^d : |\widehat{\mu}(\varrho)| > T^\kappa\}$$

is covered by $C_\varepsilon T^\varepsilon$ cubes $[0, 1)^d + \mathbf{n} \subset [-T, T]^d$, $\mathbf{n} \in \mathbb{Z}^d$.

Average Fourier decay and L^2 flattening

If μ is a measure on $\mathbb{R}^d$ and $\mathcal{D}_k = \{2^{-k}([0,1)^d + \mathbf{n}) : \mathbf{n} \in \mathbb{Z}^d\}$, define:

$$\mu_k := \sum_{D \in \mathcal{D}_k} \mu(D)\delta_D \quad \text{and} \quad \|\mu_k\|_2 := \Big(\sum_{D \in \mathcal{D}_k} \mu_k(D)^2 \Big)^{1/2}$$

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Theorem (Khalil, 2023)

If $C, \alpha > 0$, then $\forall \varepsilon > 0$, $\exists n, k_0 \in \mathbb{N}$ s.t. for all μ (C,α)-uniformly affinely non-concentrated and for all $k \geq k_0$ we have: $\|\mu_k^{*n}\|_2^2 \lesssim 2^{2d(n-1)-(d-\varepsilon)k}$.

- Implies growth of L^q dimensions under convolutions that was done in $d = 1$ by Rossi-Shmerkin (2019) that employed **Shmerkin's inverse theorem**.

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- Implies growth of L^q dimensions under convolutions that was done in $d = 1$ by Rossi-Shmerkin (2019) that employed **Shmerkin's inverse theorem**.
- **Implies average Fourier decay:** If $\kappa = \frac{\varepsilon}{4n+4d}$ and $2^k \leq T < 2^{k+1}$, then:

$$|\{\|\varrho\| \leq T, |\widehat{\mu}(\varrho)| > T^{-\kappa}\}| \cdot T^{2n\kappa} \leq \int_{\|\varrho\| \leq T} |\widehat{\mu}(\varrho)|^{2n} \, d\varrho = \int_{\|\varrho\| \leq T} |\widehat{\mu^{*n}}(\varrho)|^2 \, d\varrho$$

$$\lesssim T^{2d} \int \mu^{*n}(B(x, 1/T))^2 \, dx \lesssim T^{2d} 2^{-dk} \sum_{D \in \mathcal{D}_k} \mu^{*n}(D)^2 \lesssim_{d,n} T^{\varepsilon}$$

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- Let μ be a "dynamically defined" measure (e.g. stationary for an IFS)

Step 1: Averaging: Express $\widehat{\mu}(\xi)$ as **average** over $\widehat{\mu_x}(\xi_w)$, where
- μ_x, $x \in X$, are **blow-ups** (scaled copies of μ at pieces of $\operatorname{supp}\mu$) and
- $\xi_w \in \mathbb{R}^d$ are **random frequencies** sampled by the dynamics

Step 2: Flattening: Show $\forall x$ that $\widehat{\mu_x}$ has desired decay rate on **large set of frequencies** with exceptional set E_ξ independent of x

Step 3: Separation: Use some Diophantine, autosimilarity or non-linearity property of the dynamics s.t. ξ_w are **spread out** so they must have **small intersection** with the exceptional set E_ξ from Step 2.

Consequences

As L^2 flattening was proved in $\mathbb{R}^d$, it removes many difficulties that arise in higher dimensional theory giving also new unified proofs that avoid earlier techniques such as discretised sum-product- or renewal theory.

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As L^2 flattening was proved in $\mathbb{R}^d$, it removes many difficulties that arise in higher dimensional theory giving also new unified proofs that avoid earlier techniques such as discretised sum-product- or renewal theory. Results include:

- Polylogarithmic Fourier decay for **Diophantine self-similar** μ on $\mathbb{R}^d$
- Polynomial Fourier decay for **Patterson-Sullivan measures** μ on convex co-compact hyperbolic manifolds with Zariski dense group of isometries
- Polynomial Fourier decay for **Gibbs measures** μ **for non-linear conformal** C^2 **IFSs** in $\mathbb{R}^d$
- Polynomial Fourier decay for the stationary measures μ of carpet **non-conformal IFSs** that are non-linear in each principal direction
- New bound for **essential spectral gap** for the scattering resonances of the Laplacian and **Fractal Uncertainty Principles independent** of the doubling constant of the PS measure
- **Equidistribution of vectors** on the supports of the above μ

Some earlier works: Bourgain-Dyatlov '16, Li-Naud-Pan '21, Li-S. '21, Algom-Hertz-Wang '21, Lindenstrauss-Varjú '16, Dayan-Ganguly-Weiss '20, S.-Stevens '22 and Algom-Hertz-Wang '22, Backus-Leng-Tao '23, ...

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- Consider IFS $\{f_0(x) = r_0 x + t_0, f_1(x) = r_1 x + t_1\}$, $r_0, r_1 > 0$, s.t. $\frac{\log r_0}{\log r_1}$ is **Diophantine**: $\exists c > 0, l > 2$ s.t. $\forall \frac{p}{q} \in \mathbb{Q}$:

$$\Big|\frac{\log r_0}{\log r_1} - \frac{p}{q}\Big| > \frac{c}{q^l}.$$

(e.g. $r_0 = 1/2$ and $r_1 = 1/3$)

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  (e.g. $r_0 = 1/2$ and $r_1 = 1/3$)
- Let μ be the self-similar measure

$$\mu = \frac{1}{2}(f_0\mu + f_1\mu).$$

- Thus iterating this for any $\mathcal{W} \subset \{0,1\}^*$ s.t. cylinders $[w]$, $w \in \mathcal{W}$, form a **partition** of $\{0,1\}^{\mathbb{N}}$, we have:

$$\int \varphi(x)\, d\mu(x) = \sum_{w \in \mathcal{W}} 2^{-|w|} \int \varphi(f_w(x))\, d\mu(x)$$

  $\forall$ continuous $\varphi : [0,1] \to \mathbb{C}$ if $f_w(x) := f_{w_1} \circ .. \circ f_{w_k}(x)$.

Step 1: Averaging

- Let $\xi \in \mathbb{R}$, $|\xi| \gg 1$
- By self-similarity $f_w(x) = r_w x + t_w$ for $r_w := r_{w_1} \dots r_{w_{\log|\xi|}}$:

$$|\widehat{\mu}(\xi)| = \Big| \sum_{|r_w \xi| \sim (\log|\xi|)^{2l}} 2^{-|w|} \int e^{-2\pi i \xi f_w(x)} \, d\mu(x) \Big|$$
$$\leq \sum_{|r_w \xi| \sim (\log|\xi|)^{2l}} 2^{-|w|} |\widehat{\mu}(r_w \xi)|$$

(recall $l > 2$ comes from the Diophantine condition)

(So in the general strategy $\mu_x = \mu$ and $\xi_w = r_w \xi$)

Step 2: Flattening

- If $T = (\log|\xi|)^{2l}$ and $\varepsilon > 0$ small, let $E_\xi \subset \mathbb{Z}$ be the set of **exceptional** $n \in \mathbb{Z}$ s.t. $\exists \varrho \in [n, n+1) \cap [-T, T]$ satisfying

$$|\widehat{\mu}(\varrho)| > T^{-\kappa(\varepsilon)}.$$

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- We can reduce to sum over E_ξ using $\|\widehat{\mu}\|_\infty \leq 1$:

$$\begin{aligned}
|\widehat{\mu}(\xi)| &\lesssim \sum_{|r_w\xi| \sim (\log|\xi|)^{2l}} 2^{-|w|}|\widehat{\mu}(r_w\xi)| \\
&\lesssim \sum_{n\in\mathbb{N}} \sum_{|r_w\xi| \sim (\log|\xi|)^{2l}} 2^{-|w|}|\widehat{\mu}(r_w\xi)|\mathbf{1}(r_w\xi \in [n, n+1)) \\
&\lesssim \sum_{n\in E_\xi} \sum_{|r_w\xi| \sim (\log|\xi|)^{2l}} 2^{-|w|}\mathbf{1}(r_w\xi \in [n, n+1)) + (\log|\xi|)^{-2l\kappa(\varepsilon)}.
\end{aligned}$$

Step 3: Separation

- If $|r_w\xi|, |r_v\xi| \sim (\log|\xi|)^{2l}$ and $r_w \neq r_v$, by letting $p, q \in \mathbb{Z}$ as the **differences** of 1:s and 0:s between the words w and v, then

$$|r_w\xi - r_v\xi| = |r_v\xi|\left|1 - \frac{r_w}{r_v}\right| \geq |r_v\xi|(q\log r_1)\left|\frac{\log r_0}{\log r_1} - \frac{p}{q}\right| \gtrsim (\log|\xi|)^l > 1.$$

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- Thus $\forall n \in \mathbb{Z}$, $\exists$ unique $N_n \in \mathbb{N}$ s.t. $\forall w$ with $r_w\xi \in [n, n+1)$ s.t. $r_w = r_0^{N_n} r_1^{|w| - N_n}$.

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- **Thus**, as $|w| \gtrsim (\log|\xi|)^{c_1}$ for some $c_1 > 0$, we arrive to:

$$|\widehat{\mu}(\xi)| \lesssim \sum_{n\in E_\xi} \sum_{|r_w\xi|\sim(\log|\xi|)^{2l}} 2^{-|w|}\mathbf{1}(r_w\xi \in [n, n+1)) + (\log|\xi|)^{-2l\kappa(\varepsilon)}$$

$$\lesssim \sum_{n\in E_\xi} \max_{|r_w\xi|\sim(\log|\xi|)^{2l}} 2^{-|w|}\binom{|w|}{N_n} + (\log|\xi|)^{-2l\kappa(\varepsilon)}$$

$$\lesssim \sharp E_\xi (\log|\xi|)^{-\frac{c_1}{2}} + (\log|\xi|)^{-2l\kappa(\varepsilon)}$$

$$\lesssim (\log|\xi|)^{-c_2}$$

since $\binom{L}{N} \lesssim 2^L L^{-\frac{1}{2}}$, $\forall N \leq L$ and $\sharp E_\xi \lesssim (\log|\xi|)^\varepsilon$ and ε is small.